Lecture 2

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2. PRINCIPLES OF CIRCUIT TOPOLOGY

2.1. Basic Topology Concepts

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In circuit analysis is often used geometric representation of circuits. Such representation is based on topology. Topology is the part of mathematics that is referred to as graph theory. Topology studies the properties of geometric figures not dependent on their size [1, 4, 5].

The basic topology concepts are: branch, node, path, loop, graph, tree, edge, chord, and cross-section.

A branch is a subcircuit carrying the same current. Graphically, it is represented by a line. Fig. 2.1 shows an electric circuit diagram. The subcircuits with the resistances $r_1, r_2, r_3, r_4, r_5, r_6$, the EMF e_1, and the current source j_1 are branches, i.e. each subcircuit carries the same current $i_1, i_2, i_3, i_4, i_5, i_6, \cancel{i_2}, j_1$ respectively.

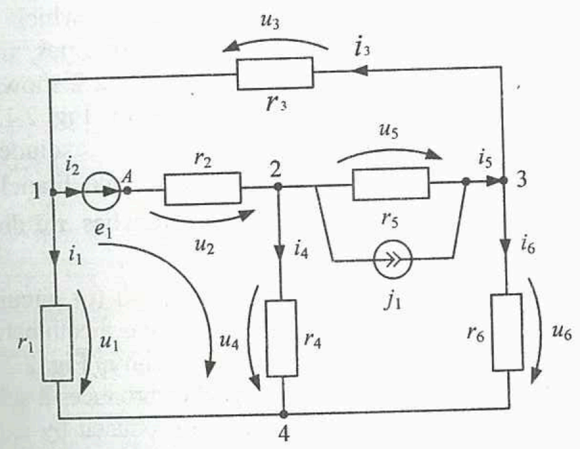

Fig. 2.1

A node is a place where branches are connected. Graphically, it is represented by a point. In Fig. 2.1 points *1–4* and A are the nodes. An eliminable node is the place where two branches connect because the common branches carry the same current and they can be replaced by one branch. The point A in Fig. 2.1 is an eliminable node because the voltage source e_1 and resistance r_2 carry the same current i_2 and these elements can be represented by one branch. All the nodes of the diagram, except one, are called independent.

A path is a set of branches connecting two nodes without branching. In Fig. 2.1 the brunches $e_1 - r_2, r_3 - r_5, r_1 - r_4, r_1 - r_6 - r_5, r_3 - r_6 - r_4$ make paths between nodes *1* and *2*. The branches $r_1 - r_4 - r_6 - r_5$ do not make a path because they have a branching in the section $r_4 - r_6 - r_5$.

A loop is a closed path encompassing several branches. In Fig. 2.1 the branches $r_1 - e_1 - r_2 - r_4$, $r_4 - r_6 - r_5$, $r_2 - e_1 - r_3 - r_5$, $r_1 - r_3 - r_6$, $r_1 - r_3 - r_5 - r_4$, $r_1 - e_1 - r_2 - r_5 - r_6$, $r_4 - r_2 - e_1 - r_3 - r_6$, make loops. An eliminable loop is a loop that includes a branch with an ideal current source because the internal resistance of an ideal current source is infinitely great, and, in terms of resistance, a branch with an ideal current source is equivalent to a break. In addition, the current of an ideal current source is a known and unvariable value. In Fig. 2.1, the loop $r_5 - j_1$ is eliminable.

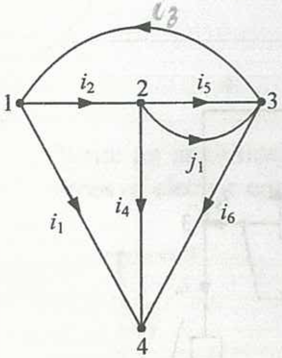

Fig. 2.2

A graph is a geometric representation of an electric circuit in which all circuit branches are replaced by lines, and nodes are replaced by points. Fig. 2.2 shows a graph of the electric circuit in Fig. 2.1. Here the eliminable node A is excluded and the elements e_1, r_2 make one branch i_2 of the graph. The other branches are designated as $i_1, i_3, i_4, i_5, i_6, j_1$.

The connected graph (or circuit connected graph) is a graph that has a path between any of its two nodes. The graph in Fig. 2.2 is a circuit-connected graph. Excluding some graph branches produces a subgraph. The subgraph in Fig. 2.2 is represented as a graph produced by excluding, for example, branches i_5, j_1 and, thus, including only branches i_1, i_2, i_3, i_4, i_6.

A graph tree is a subgraph of a circuit-connected graph that includes all graph nodes but contains no loop. Some trees of the graph in Fig. 2.2 are shown in Fig. 2.3 (solid lines).

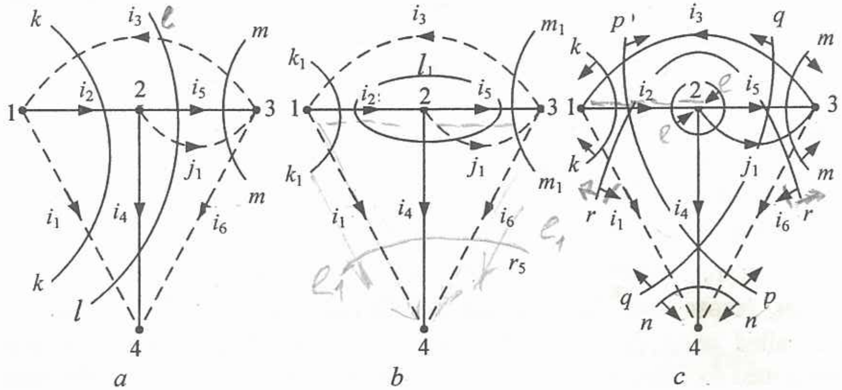

Fig. 2.3

Apparently, it is possible to build several trees for the given graph because graph nodes can be connected in different ways. However, a graph tree cannot include branches with an ideal current source for the reason indicated earlier, i.e. when building a graph tree, any ideal current source is replaced by a break.

Thus, the number of graph tree branches is one less than the number of nodes, that is it equals the number of independent graph nodes.

An edge is a branch of a graph tree. In Fig. 2.3, *a* the branches i_2, i_4, i_5 are edges. In Fig. 2.3, *b* the branches $i_1, i_4 - i_6$ are defined as edges. In Fig. 2.3, *b* the branches i_3, i_4, i_5 are edges.

A subgraph is called a complementary to a tree when it complements a tree to a graph. Some tree complements (dotted lines) are shown in Fig. 2.3.

A chord is a branch that does not belong to a tree. In Fig. 2.3, *a* the branches i_1, i_3, i_6 are chords of the tree made by the branches i_2, i_4, i_5; in Fig. 2.3, *b* the branches i_2, i_3, i_5 are the chords of the tree made by the branches i_1, i_4, i_6. In Fig. 2.3, *b* branches i_1, i_2, i_6 are the chords of the tree made by the branches i_3, i_4, i_5. A branch made by ideal the ideal current source *j* cannot be a chord. A loop is made by adding a chord to a tree. Such a loop is the main loop or an independent loop.

In Fig. 2.3, *a* the main loops are $i_1 - i_2 - i_4$, $i_2 - i_3 - i_5$, $i_4 - i_5 - i_6$. In Fig. 2.3, *b* the main loops are $i_1 - i_2 - i_4$, $i_1 - i_3 - i_6$, $i_4 - i_5 - i_6$. In Fig. 2.3, *c* the main loops are $i_1 - i_3 - i_4 - i_5$, $i_2 - i_3 - i_5$, $i_4 - i_5 - i_6$. The number of main loops equals the number of graph tree chords.

A cross-section is a set of circuit-connected graph branches, the elimination of which (but not the endings of the branch set) makes a graph which is not a circuit connected one. In order to obtain a cross-section we use a section line (or surface), while none of the branches is crossed twice. A line or a surface of a section divides a graph into two parts. Fig. 2.3, *a* shows the sections $i_1 - i_2 - i_3$, $i_1 - i_3 - i_4 - i_5 - i_1$, $i_3 - i_5 - i_6 - j_1$ obtained by means of the section lines $k - k$, $l - l$, $m - m$, respectively. Sections that include only one ребро edge of a selected tree are called main or independent sections. Proceeding from the above-mentioned sections, we can say that the sections $k - k$, $m - m$ are the main ones, i.e. the first of them includes only one edge i_2, and the second — only one edge i_5 of the graph tree. The section $l - l$ is not the main one because it includes two edges i_4, i_5 of the graph tree. In Fig. 2.3, *b* all sections $k_1 - k_1$, $l_1 - l_1$, $m_1 - m_1$ are the main ones, because each of them includes only one edge i_1, i_4, i_6 of the graph tree, respectively.

2.2. Topological Matrices

For analytical description of electric circuit graphs and their storage in computer memory in digital form, it is more convenient to represent graphs in the form of topological matrices. There are incidence matrices (node matrices), loop matrices and graph section matrices.

2.2.1. Incidence Matrices

It is said that if a node *i* is the end of a branch *j*, then they are incident. Information contained in a directed graph can be fully represented by a matrix called an incidence matrix (node matrix).

An $n \times b$ matrix is called an incidence matrix A_a that corresponds to a directed graph with *n* nodes and *b* branches

$$A_a = \left[a_{ij}\right],$$

where a_{ij} is an element of the matrix A_a; $a_{ij} = 1$ if the branch *j* is incident to the node *i* and directed from the node; $a_{ij} = -1$ if the branch *j* is incident

to the node *i* and directed to the node; $a_{ij} = 0$ if the branch *j* is not incident to the node *i*. For instance, we will obtain the matrix (2.1) for the directed graph shown in Fig. 2.3, *c*. It's clear from the matrix that the number of non-zero elements in each line of the matrix A_a is equal to the number of branches incident to the corresponding node. Each column contains only

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & j_1 \end{array} \leftarrow \text{branches} \\ \left[\begin{array}{ccccccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 \end{array}\right] \end{array} \underbrace{\begin{array}{llll} i_1, & i_2, & i_3 & \\ i_2, & i_4, & i_5, & j_1 \\ i_3, & i_5, & i_6, & j_1 \\ i_1, & i_4, & i_6 & \end{array}}_{s} \qquad (2.1)$$

incident branches

two non-zero elements: "+1" and "–1" because each branch is incident to two nodes and directed from one of them to the other. The sum of all elements of each column and, consequently, the sum of all matrix A_a lines is equal to zero, i.e. the matrix A_a lines are linearly dependent. Therefore, it's possible to exclude any line of the matrix A_a without any information loss. So, when the 4-th line is excluded in (2.1) we get:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & j_1 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 \end{array}\right] \end{array}.$$

Matrix *A* is called a reduced incidence matrix.

2.2.2. The Loop Matrix

Topological matrices can be derived for circuit loops too.

$$B_a = \left[b_{ij}\right],$$

where, b_{ij} are elements of the matrix B_a. $b_{ij} = 1$ if the branch *j* is incident to the loop *i* and coincides with the direction of loop path-tracing; $b_{ij} = -1$ if the branch *j* is incident to the loop *i* and opposite to the direction of loop path tracing; $b_{ij}=0$ if the branch j is not incident to the loop i. The loop matrix B_a corresponding to a directed graph with n_b loops and b branches refers to a matrix $n_b \times b$. For instance, for the directed graph in Fig. 2.3, c we will get the matrix (2.2) when loop path-tracing is clockwise. Here, the branch of the current source j_1 is not used in the matrix because such a branch does not form a separate loop as mentioned above.

It's obvious that the parts of loops not used in the matrix (2.2) are linearly dependent. A loop that includes at least one branch not being part of any other loops is linearly independent.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \leftarrow \text{branches} \end{array} \\
B_a = & \left[\begin{array}{cccccc}
-1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 & 1 \\
-1 & 0 & -1 & 1 & -1 & 0 \\
0 & -1 & -1 & 0 & -1 & 0 \\
0 & -1 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right]
\underbrace{\begin{array}{l}
i_1 - i_2 - i_4 \\
i_1 - i_2 - i_5 - i_6 \\
i_1 - i_3 - i_6 \\
i_1 - i_3 - i_4 - i_5 \\
i_2 - i_3 - i_5 \\
i_2 - i_3 - i_4 - i_6 \\
i_4 - i_5 - i_6
\end{array}}_{\text{loops}}
\end{array}
\qquad (2.2)
$$

For the planar circuits it's easy to define the number of independent loops if we take these loops as meshes of the graph grid or "windows". So for the graph in Fig. 2.3, c the loops will be the following: $i_1-i_2-i_4$, $i_2-i_3-i_5$, $i_4-i_5-i_6$. Then we get the submatrix B_b from the matrix B_a. grid – решетка

$$
\begin{array}{cc}
 & \begin{array}{cccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \end{array} \\
B_b = & \left[\begin{array}{cccccc}
-1 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right]
\begin{array}{l}
i_1 - i_2 - i_4 \\
i_2 - i_3 - i_5 \\
i_4 - i_5 - i_6
\end{array}.
\end{array}
$$

The matrix B_b is called the matrix of the main loops.

For nonplanar circuits, it is difficult to define the number of dependent loops from the number of "windows" in the graph. In this case, the T-graph

подсоединением к дереву
tree is used. A loop is formed by the connection of any chord to a tree. This loop is called the main loop. The direction of main loop path-tracing is taken as the chord direction. Thus, the number of the main loops is equal to the number of chords of the selected graph tree. It is evident that it is equal to $b-n+1$. So, the main loops for the tree in Fig. 2.3, ~~b~~ c are formed by the chords: chord i_1-loop $i_1-i_3-i_4-i_5$, chord i_2-loop $i_2-i_3-i_5$, chord i_6-loop $i_4-i_5-i_6$. Having arranged the edges in the lower columns, we get the matrix of the main loops B for the graph with the tree in Fig. 2.3, ~~b~~ c

расположить

$$
\begin{array}{cc}
 & \overbrace{i_3 - i_4 - i_5}^{\text{Tree}}\ \overbrace{i_1 - i_2 - i_6}^{\text{chords}} \\
B = & \left[\begin{array}{ccc|ccc}
1 & -1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 1
\end{array}\right]
\begin{array}{l} i_1 \\ i_2 \\ i_6 \end{array}.
\end{array}
\qquad (2.3)
$$

As seen from (2.3), any matrix B can be divided in the following way:

$$B=[B_T \mid 1],$$

where matrix B_T corresponds to the edges

$$
\begin{array}{cc}
 & \begin{array}{ccc} i_3 & i_4 & i_5 \leftarrow \text{edges} \end{array} \\
B_T = & \left[\begin{array}{ccc}
1 & -1 & 1 \\
1 & 0 & 1 \\
0 & -1 & 1
\end{array}\right]
\begin{array}{l} i_1 \\ i_2 \\ i_6 \end{array}. \\
 & \qquad\qquad \uparrow \text{ chords}
\end{array}
$$

Matrix 1 is a unit matrix and corresponds to the chords

$$
\begin{array}{cc}
 & \begin{array}{ccc} i_1 & i_2 & i_3 \leftarrow \text{edges} \end{array} \\
1 = & \left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
\begin{array}{l} i_1 \\ i_2 \\ i_6 \end{array}. \\
 & \qquad\qquad \uparrow \text{ chords}
\end{array}
$$

So, because the unit matrix is present in the matrix B, we can say that the lines of the matrix B are linearly independent.

2.2.3. The Section Matrix

Topological matrixes can be derived (получена) for graph sections too.

A section matrix D_a corresponding to a directed graph with n_c sections and b branches refers (относится) to a matrix $n_c \times b$.

$$D_a = \left[d_{ij} \right],$$

where d_{ij} is an element of the matrix D_a, $d_{ij} = 1$, if the branch j is incident to the section i and coincides with the direction of the section; $d_{ij} = -1$, if the branch j is incident to the section i and opposite to the direction of the section; $d_{ij} = 0$, if the branch j is not incident to the section i. For instance, for the directed graph in Fig. 2.2, c it is possible to choose 7 sections: $k-k$, $l-l$, $m-m$, $n-n$, $p-p$, $q-q$, $r-r$. Section directions are marked with arrows (стрелка) on the section lines. As a result, we get a section matrix

$$\begin{array}{cc} & \begin{array}{ccccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & j_1 & \leftarrow \text{branches} \end{array} \\ D_a = & \left[\begin{array}{ccccccc} -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \begin{array}{l} k-k \\ l-l \\ m-m \\ n-n \\ p-p \\ q-q \\ r-r \end{array} . \\ & \qquad\qquad\qquad\qquad \uparrow \text{ sections} \end{array} \tag{2.4}$$

It is obvious that some sections used in the matrix (2.4) are linearly dependent. Only a section that includes at least one branch not being part of any other section is considered to be linearly independent. The number of independent sections, evidently, is equal to the number of independent nodes. Therefore, it is possible to retain any three lines in the matrix (2.4) without losing any information. So, having excluded the first four lines, we get the matrix:

$$D_b = \begin{array}{c} \begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & j_1 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \begin{array}{l} p-p \\ q-q \\ r-r \end{array} \end{array} .$$

The matrix D_b is called the base section matrix. A systematic method of building the base section matrix involves the use of the tree T as in this case the base matrix will correspond to the graph main sections. Such a matrix is called the main section matrix D. The directions of the main sections are taken according to the direction of the corresponding edges of the graph tree. So, having arranged the edges in the lower columns, we get the matrix D:

$$D = \begin{array}{c} \begin{array}{cc} \overbrace{\begin{array}{ccc} i_3 & i_4 & i_5 \end{array}}^{\text{edges (рёбра)}} & \overbrace{\begin{array}{cccc} i_1 & i_2 & i_6 & j_1 \end{array}}^{\text{the rest of the graph branches}} \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} i_3 \;\; k-k \\ i_4 \;\; n-n \\ i_5 \;\; N\;N \end{array} \\ \qquad\qquad\qquad \uparrow \text{ edges} \end{array} \tag{2.5}$$

One can see from (2.5) that any matrix D can be divided in the following way:

$$D = \left[1 \middle| D_L \right],$$

where the unit matrix 1 corresponds to the edges

$$1 = \begin{array}{c} \begin{array}{cccc} i_3 & i_4 & i_5 & \leftarrow \text{edges} \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{array}{l} i_3 \\ i_4 \\ i_5 \end{array} . \\ \qquad \uparrow \text{ edges} \end{array}$$

The matrix D_L corresponds to the rest of the graph branches

$$D_L = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|l} i_1 & i_2 & i_6 & j_1 & \leftarrow \text{the rest of the graph branches} \\ \end{array}$$

$$D_L = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{array}{l} i_3 \\ i_4 \\ i_5 \end{array}.$$

$\uparrow$ edges

As the matrix D has the unit matrix in itself, we can say that the lines of the matrix D are linearly independent.

Example 1

Draw a directed graph of the circuit (Fig. 2.4), choose one of the trees of the graph and do for this tree the following:

a) make a system of the main cross-sections;

b) make a system of the main loops.

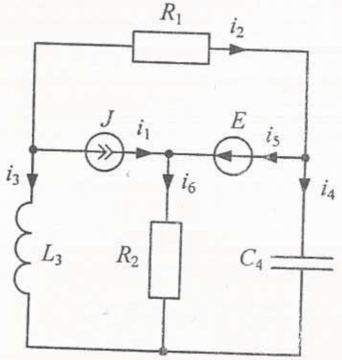

Fig. 2.4

The graph of the electric circuit is obtained by replacement of all branches of the circuit with line segments. The resultant graph is shown in Fig. 2.5, *a*. If directions are indicated on the branches, the graph is directed (Fig. 2.5, *b*). Here and hereafter figures are used to number nodes and to mark currents in the branches. In Fig. 2.5, *c* one of the possible graph trees is shown. In Fig. 2.5, *d* is depicted a system of cross-sections with regard to the selected tree. The first section C_1 contains only one (the first) branch of the tree and two chords — the fourth and the sixth. Section C1 divides the graph into two parts. One part consists of the branches i_2, i_3, i_5 with nodes 1, 2, 4, and the other — only from node 3. Cross-section C_2 contains only one (the second) branch of the tree and two chords — the fifth and the sixth. This section also divides the graph into two parts. One part consists of the branches i_1, i_3, i_4 with nodes 1, 3, 4, and the other — only from node 2. The last cross section C_3 of this system of the main cross sections contains only one (the third) branch and three chords — the fourth, the fifth and the sixth. This cross-section divides the graph into two parts.

One part consists of the branch i_1 with nodes 1 and 3, and the other — the branch i_2 with nodes 2 and 4. Note that all sections differ from each other by at least one branch, namely, the branch of the tree.

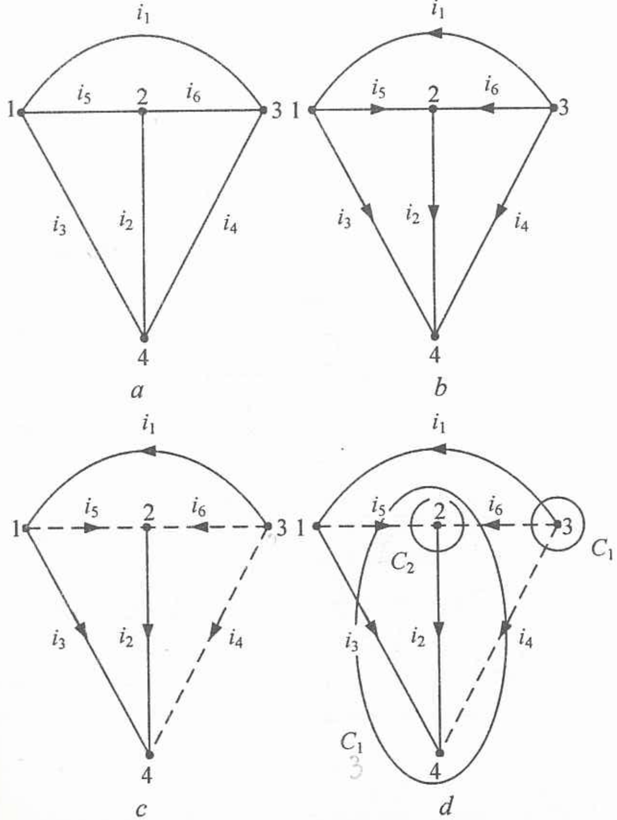

Fig. 2.5

The system of loops on the selected tree is built in the following way. We add one of the chords (for example, the fourth) to the selected tree (Fig. 2.5, *c*). The result is a loop consisting of the tree branches i_1, i_3 and the chord i_4. Then we add another chord, for example, i_5. The loop obtained as a result of such addition consists of the tree branches i_2, i_3 and the chord i_5. The last loop of the system of the main loops on the selected tree is obtained by adding the sixth chord i_6. Note that all paths are different from each other in at least one branch. This branch is one of the chords — i_4.

Questions for self-checking

1. Give definitions of the main topological concepts (branch, node, path, loop, tree, edge, chord, section). Specify these concepts in the electric diagram.

2. State the rules for constructing the matrix of incidences (nodes), the matrix of loops, and the matrix of cross-sections. How are the main loop and its direction determined? How are the main section and its direction determined?

Problems

1. Construct a graph of the electric circuit whose diagram is shown in Fig. 2.6.

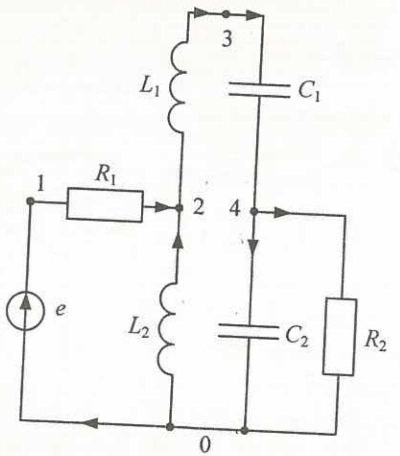

Fig. 2.6

2. Construct a graph of the electric circuit for the incidence matrix:

$$A=\begin{vmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \end{vmatrix}.$$

3. Build several different trees of the graph in Fig. 2.7. Point out the system of main loops, corresponding to each tree.

4. Derive a matrix of the main sections of the graph shown in Fig. 2.7. Specify the tree corresponding to this section and composed of branches 1, 2, 4, 8.

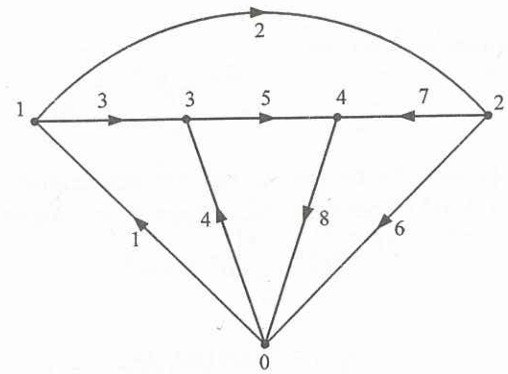

Fig. 2.7

5. Derive a matrix of the main loops of the circuit using the initial data of problem 4.

3. HARMONIC CURRENTS AND THEIR DESCRIPTION USING COMPLEX NUMBERS

3.1. Harmonic Currents, Their Characteristics, Basic Concepts and Definitions

Electric current can be direct (DC) and alternating (AC).

Direct refers to current the value of which does not change in time.

If the values of current are different in different instants of time, the current is called alternating. An instantaneous current value is designated as $i(t)$. In a similar way are designated an instantaneous value of voltage $u(t)$, that of EMF $e(t)$ and others [6–8].

Periodic refers to AC for which the relation is observed:

$$i(t)=i(t+T),$$

where T is the period — minimum time interval after which the current values begin to repeat themselves. T is measured in seconds (s).

The number of periods per second is called frequency f

$$f=\frac{1}{T}.$$

Frequency is measured in Hertz (Hz).

Angular frequency is determined by the expression

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Angular frequency is measured in radians per second (rad/s).

A widespread kind of periodic AC is harmonic current

широко распространен.

$$i(t) = I_m \sin(\omega t + \psi) \tag{3.1}$$

or

$$i(t) = I_m \cos(\omega t + \psi). \tag{3.2}$$

Here I_m is the amplitude (maximum current), $\theta = \omega t + \psi$ — oscillation phase, measured in radians (rad), ψ — the initial phase, i.e. the value of θ for $t = 0$.

From (3.1) and (3.2) with $\omega = 0$, we obtain a direct current. That's why the frequency of DC is said to be equal to zero.

Fig. 3.1 shows a graph of the harmonic current (3.1).

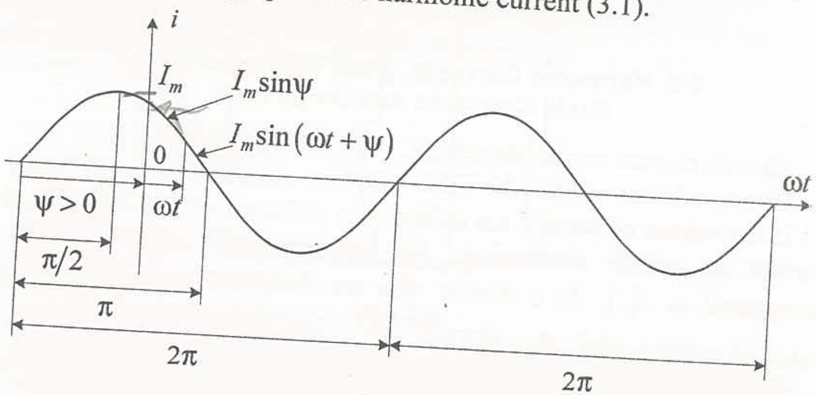

Fig. 3.1

The industrial frequency of harmonic current varies. In Western Europe the industrial frequency is 50 Hz, in the USA — 60 Hz. In geologic exploration currents are used whose frequencies constitute fractions of a Hz, whereas in radio they reach up to 30 GHz.

исследованиях составляют

The main properties of harmonic current are:

– harmonic current's derivative of any order of is harmonic current,

– the sum of any number of harmonic currents of a certain frequency is harmonic current of the same frequency.

3.2. The Effective and Mean Values of a Harmonic Current

The effective value of a harmonic current is its root-mean-square value for the period

$$I = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2(\omega t + \psi)\,dt} =$$

$$= I_m \sqrt{\frac{1}{T}\int_0^T \frac{1 - \cos\left[2(\omega t + \psi)\right]}{2}\,dt} = \tag{3.3}$$

$$= I_m \sqrt{\frac{1}{2T}\cdot T - \frac{1}{2T}\int_0^T \cos\left[2(\omega t + \psi)\right]dt} = \frac{I_m}{\sqrt{2}} = 0{,}707 I_m.$$

Squaring the left and right parts in (3.3) and multiplying them by a resistance r we get

$$I^2 r T = \int_0^T i^2(t)\, r\,dt = w(T).$$

From here we can see that the effective value of the harmonic current I is equal to the value of the direct current I that releases the same energy on the resistance r for a period T as the harmonic current.

The mean (half-period average) value of the harmonic current is defined in the positive half-period

$$I_{av} = \frac{1}{\frac{T}{2}} \int_{-\frac{\psi}{\omega}}^{\frac{T}{2}-\frac{\psi}{\omega}} i(t)\,dt = \frac{2}{T}\int_{-\frac{\psi}{\omega}}^{\frac{T}{2}-\frac{\psi}{\omega}} I_m \sin(\omega t + \psi)\,dt = \tag{3.4}$$

$$= \frac{2I_m}{T\omega}\left[-\cos(\omega t + \psi)\right]\Bigg|_{-\frac{\psi}{\omega}}^{\frac{T}{2}-\frac{\psi}{\omega}} = \frac{2I_m}{2\pi f T}(-\cos\pi + \cos 0) = \frac{2I_m}{\pi} = 0.637 I_m.$$

Analogical expressions for the effective and mean values can be written for voltages and EMF

$$U = \frac{U_m}{\sqrt{2}} = 0{,}707 U_m; \quad E = \frac{E_m}{\sqrt{2}} = 0{,}707 U_m;$$

$$U_{av} = \frac{2U_m}{\pi} = 0{,}637 U_m; \quad E_{av} = \frac{2E_m}{\pi} = 0{,}637 U_m.$$

It's obvious from (3.3), (3.4) that

$$I=\frac{I_m}{\sqrt{2}}=\frac{\pi I_{av}}{2\sqrt{2}}=1{,}11 I_{av};$$

$$I_{av}=\frac{2I_m}{\pi}=\frac{2\sqrt{2}I}{\pi}=0{,}9I.$$

A3.

3.3. Harmonic Current Representation by Means of Complex Values

It is known that any complex number

$$\dot{A}=a+jb \tag{3.5}$$

can be represented on a complex plane as a point. The complex plane can be represented as a Cartesian (rectangular) coordinate system in which the real part of the complex number $Re\,A = a$ is plotted on the x-axis; and its imaginary part $Im\,A = b$ — on the y-axis (Fig 3.2).

The complex number representation (3.5) is called algebraic. From Fig. 3.2 it follows

$$A=\sqrt{a^2+b^2},\quad \alpha=\operatorname{atan}\frac{b}{a},$$

$$a=A\cos\alpha,\quad b=A\sin\alpha$$

that's why

$$\dot{A}=A\cos\alpha+jA\sin\alpha.$$

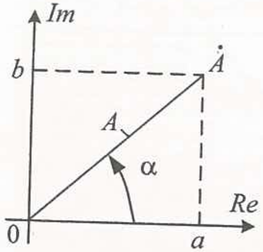

Fig 3.2

Such representation of a complex number is called trigonometric.

According to Euler's formula:

$$\cos\alpha+j\sin\alpha=e^{ja}. \tag{3.6}$$

Therefore,

$$\dot{A}=Ae^{ja}. \tag{3.7}$$

It is the exponential representation.

It is obvious that using (3.6) and (3.7) we can write:

$$I_m e^{j(\omega t+\psi)}=I_m\cos(\omega t+\psi)+jI_m\sin(\omega t+\psi);$$

$$Re\left[I_m e^{j(\omega t+\psi)}\right]=I_m\cos(\omega t+\psi) \tag{3.8}$$

(3.8) describes the cosinusoidal current of (3.5), but the imaginary part

$$Im\left[I_m e^{j(\omega t+\psi)}\right]=I_m\sin(\omega t+\psi)$$

describes the sinusoidal current of (3.4).

Thus, a harmonic current can be represented by a complex number, that is the sinusoidal (3.4) and cosinusoidal (3.5) currents can be represented by a single complex number

$$Im\left[I_m e^{j(\omega t+\psi)}\right]=I_m\sin(\omega t+\psi). = \dot{I}_m(t)$$

The value $\dot{I}_m(t)$ is called the complex instantaneous harmonic current. It can be rewritten in the following way:

$$\dot{I}_m(t)=I_m e^{j(\omega t+\psi)}=I_m e^{j\psi}e^{j\omega t}=\dot{I}_m e^{j\omega t}.$$

The value

$$\dot{I}_m=I_m e^{j\psi} \tag{3.9}$$

is called the complex amplitude of a current. The complex effective value of a current is written as

$$\dot{I}=I_m e^{j\psi}.$$

The function $e^{j\omega t}$ is called the rotation operator. It shows that the angle (phase) $\theta=\omega t+\psi$ increases as time t increases, and the vector $\dot{I}_m(t)$ has a counter-clockwise rotation round the point 0 on the complex plane.

As a result, the harmonic currents $i(t)=I_m\sin(\omega t+\psi)$ or $i(t)=I_m\cos(\omega t+\psi)$ on the complex plane are defined completely by the complex amplitude $\dot{I}_m$.

The value $\dot{I}_m$ is called the image and $i(t)$ — the original of a harmonic current.

The image in terms of the complex amplitude $\dot{I}_m$ from the original in terms of the real value $i(t)$ is obtained from the formula of the direct K-conversion.

$$K\left[i(t)\right]=\frac{2}{T}\int_0^T I(t)e^{-j\omega t}dt. \tag{3.10}$$

So, from (3.10), for the harmonic current (3.5) we get:

$$K\left[i(t)\right]=\frac{2}{T}\int_0^T I_m\cos(\omega t+\psi)e^{-j\omega t}dt=$$

$$=\frac{2I_m}{T}\int_0^T\frac{e^{j(\omega t+\psi)}+e^{-j(\omega t+\psi)}}{2}e^{-j\omega t}dt=\frac{I_m}{T}\left[\int_0^T e^{j\psi}dt+\int_0^T e^{-j(2\omega t+\psi)}dt\right]=$$

$$=\frac{I_m}{T}\left\{Te^{j\psi}-\frac{1}{2j\omega}\left[e^{-j(4\pi+\psi)}-e^{-j\psi}\right]\right\}=I_m e^{j\psi},$$

which corresponds to (3.9).

The original — a real harmonic function $i(t)$ — is obtained from its image $K\left[i(t)\right]=\dot{I}_m=I_m e^{-j\psi}$ by the formula of the inverse K-conversion

$$i(t)=\frac{1}{2}\left(\dot{I}_m e^{j\omega t}+\overset{*}{I}_m e^{-j\omega t}\right),$$

where

$$\overset{*}{I}_m=I_m e^{-j\psi} \tag{3.11}$$

is the complex conjugate amplitude of the current. Thus, from (3.11), for the complex amplitude (3.9) we get:

$$i(t)=\frac{1}{2}\left(I_m e^{j\psi}e^{j\omega t}+I_m e^{-j\psi}e^{-j\omega t}\right)=$$

$$=I_m\frac{e^{j(\omega t+\psi)}+e^{-j(\omega t+\psi)}}{2}=I_m\cos(\omega t+\psi),$$

which corresponds to (3.5).

The original and the image are related as

$$i(t)\doteqdot \dot{I}_m.$$

Let us determine the image of the derivative of the harmonic current. Let $i(t)$ be defined by (3.5). Then

$$\frac{dI(t)}{dt}=\frac{d\left[I_m\cos(\omega t+\psi)\right]}{dt}=-\omega I_m\sin(\omega t+\psi). \tag{3.12}$$

Applying the direct K-conversion (3.10) to (3.12) we get:

$$K\left[\frac{di(t)}{dt}\right]=\frac{2}{T}\int_0^T\left[-\omega I_m\sin(\omega t+\psi)\right]e^{-j\omega t}dt=$$

$$=\frac{-2\omega I_m}{T}\int_0^T\frac{e^{j(\omega t+\psi)}-e^{-j(\omega t+\psi)}}{2j}e^{-j\omega t}dt=\frac{-2\pi I_m}{jT^2}\left[\int_0^T e^{j\psi}dt-\int_0^T e^{-j(2\omega t+\psi)}dt\right]=$$

$$=\frac{j2\pi I_m}{T^2}\left\{Te^{j\psi}-\frac{1}{2j\omega}\left[e^{-j(4\pi+\psi)}-e^{-j\psi}\right]\right\}=j\omega I_m e^{j\psi}=j\omega\dot{I}_m.$$

Hence, taking the derivative of the original in the real form is equivalent to complex image multiplication by the imaginary frequency $j\omega$. It is obvious that

$$K\left[\frac{di^2(t)}{dt^2}\right]=j\omega j\omega\dot{I}_m=(j\omega)^2\dot{I}_m.$$

And for the derivative of the n^{th} order:

$$K\left[\frac{di^n(t)}{dt^n}\right]=(j\omega)^n\dot{I}_m.$$

Determine the image of the integral of a harmonic current. Let $i(t)$ be defined by (3.5). Then

$$\int_0^t i(\tau)d\tau=\int_0^t I_m\cos(\omega\tau+\psi)d\tau=\frac{I_m}{\omega}\left[\sin(\omega t+\psi)-\sin\psi\right]. \tag{3.13}$$

Applying the direct K-conversion (3.10) to (3.13) we get:

$$K\left[\int_0^t i(\tau)d\tau\right]=\frac{2}{T}\int_0^T\frac{I_m}{\omega}\left[\sin(\omega t+\psi)-\sin\psi\right]e^{-j\omega t}dt=$$

$$=\frac{2I_m}{\omega T}\left[\int_0^T\frac{e^{j(\omega t+\psi)}-e^{-j(\omega t+\psi)}}{2j}e^{-j\omega t}dt-\sin\psi\int_0^T e^{-j\omega t}dt\right]=$$

$$\frac{I_m}{\pi}\left\{\frac{1}{2j}\left[\int_0^T e^{j\psi}dt-\int_0^T e^{-j(2\omega t+\psi)}dt\right]-\sin\psi\int_0^T e^{-j\omega t}dt\right\}=$$

$$=\frac{I_m}{2\pi j}e^{j\psi}T=\frac{I_m}{j\omega}e^{j\psi}=\frac{1}{j\omega}\dot{I}_m.$$

Thus, taking the integral of the original in the real form is equivalent to dividing the complex image by the imaginary frequency $j\omega$. It is obvious that

$$K\left[\int_0^t\int_0^s i(\tau)d\tau\right]=\frac{1}{j\omega}\frac{1}{j\omega}\dot{I}_m=\left(\frac{1}{j\omega}\right)^2\dot{I}_m.$$

And for an n-fold integral

$$K\Bigg[\underbrace{\int_0^t\int_0^s\ldots\int_0^r}_{n} i(\tau)\,d\tau\Bigg]=\left(\frac{1}{j\omega}\right)^n\dot{I}_m.$$